Capita
Q-configurations

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Abstract

In the first section I introduce a measuring of angles by the cotangens $\breve{a}$ of half the angle α. This gives an isomorphism between Q-angles, the rational numbers with an appropriate group-operation and the rational points on the graph of $c(x) = \frac{x^2-1}{x^2+1}$. In the second section I discuss the possibiity of the construction of Q-triangles given by three random rationals ASA, AAS, SAS, SSA and SSS. For instance AAS by $\breve{a}$, $\breve{b}$ and a. In the last section follow two Q-configurations. The first consist of 6 small triangles around the orthocenter of a triangle. The second consist of 9 small triangles around the circumcenter of a triangle.

Contents

1 Q-angles and the curve $(x^2+1)y = x^2-1$.

1.1 $\cot \frac{1}{2}\alpha = \breve{a}$

Definition.
A Q-angle is an angle of a Heron-triangle.

It is well known that these angles are also the angles of Pythagorean triangles, because the altitudes of Heron-triangles divide them into two Pythagorean triangles with sides $a = 2mn$, $b = m^2 - n^2$ and $c = m^2 + n^2$. Let $z = m + n \cdot \imath$ be complex whole number in $\mathbb{Z}[\imath]$, then $z^2 = (m^2 - n^2) + (2mn) \cdot \imath$. And so it is clear that Q-angles are the arguments of squares of complex numbers. Without loss of generality we can take a rational number $\breve{q} = \frac{m}{n}$, and this $\breve{q}$ is just the cotangens of half the Q-angle. So we come to another definition of Q-angles.

Definition.
An angle α is a Q-angle $\Leftrightarrow \cot \frac{1}{2}\alpha = \breve{a}$ with $\breve{a} \in \mathbb{Q}$

Some basicfunctions related to angles expressed in $\breve{a} \in \mathbb{Q}$.
$\sin \alpha = \frac{2 \sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha}{\cos^2 \frac{1}{2}\alpha + \sin^2 \frac{1}{2}\alpha} = \frac{2 \cot \frac{1}{2}\alpha}{\cot^2 \frac{1}{2}\alpha + 1} = \frac{2\breve{a}}{\breve{a}^2+1}$
$\cos \alpha = \frac{\cos^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\alpha}{\cos^2 \frac{1}{2}\alpha + \sin^2 \frac{1}{2}\alpha} = \frac{\cot^2 \frac{1}{2}\alpha - 1}{\cot^2 \frac{1}{2}\alpha + 1} = \frac{\breve{a}^2-1}{\breve{a}^2+1}$
$\breve{a} = \cot \frac{1}{2}\alpha = \frac{2 \cos^2 \frac{1}{2}\alpha}{2 \sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha} = \frac{1+\cos\alpha}{\sin\alpha}$
$\sin(\alpha \pm \beta) = \frac{2\breve{a}(\breve{b}^2-1) \pm 2\breve{b}(\breve{a}^2-1)}{(\breve{a}^2+1)(\breve{b}^2+1)} = \frac{2(\breve{a} \pm \breve{b})(\pm \breve{a}\breve{b} - 1)}{(\breve{a}^2+1)(\breve{b}^2+1)}$
$\cos(\alpha \pm \beta) = \frac{(\breve{a}^2-1)(\breve{b}^2-1) \mp 4\breve{a}\breve{b}}{(\breve{a}^2+1)(\breve{b}^2+1)} = \frac{(\pm \breve{a}\breve{b} - 1)^2 - (\breve{a} \pm \breve{b}^2)}{(\breve{a}^2+1)(\breve{b}^2+1)}$
$\cot \frac{1}{2}(\alpha \pm \beta) = \frac{\pm \cot \frac{1}{2}\alpha \cot \frac{1}{2}\beta - 1}{\cot \frac{1}{2}\alpha \pm \cot \frac{1}{2}\beta} = \frac{\pm \breve{a}\breve{b} - 1}{\breve{a} \pm \breve{b}}$
$\cot 45^\circ = 1 \rightarrow \sin 90^\circ = \frac{2}{2} = 1$ and $\cos 90^\circ = \frac{1-1}{1+1} = 0$
$\cot 90^\circ = 0 \rightarrow \sin 180^\circ = \frac{0}{1} = 0$ and $\cos 180^\circ = \frac{0-1}{0+1} = -1$
$\sin(180^\circ - \alpha) = \frac{2(0-\breve{a})(-0-1)}{(0+1)(\breve{a}^2+1)} = \frac{2\breve{a}}{\breve{a}^2+1}$

Proposition.
Let be given a triangle ABC and let $\breve{a}$, $\breve{b}$ and $\breve{c}$ be the cotangens of half the

angles α, β and γ respectively. Then $\breve{a} \cdot \breve{b} \cdot \breve{c} = \breve{a} + \breve{b} + \breve{c}$.
Proof.
$\cot 90^\circ = \cot \frac{1}{2}((\alpha+\beta)+\gamma) = \frac{\cot \frac{1}{2}(\alpha+\beta) \cot \frac{1}{2}\gamma - 1}{\cot \frac{1}{2}(\alpha+\beta) + \cot \frac{1}{2}\gamma} = \frac{\frac{\breve{a}\breve{b}-1}{\breve{a}+\breve{b}}\breve{c}-1}{\frac{\breve{a}\breve{b}-1}{\breve{a}+\breve{b}}+\breve{c}} = \frac{\breve{a}\breve{b}\breve{c}-\breve{a}-\breve{b}-\breve{c}}{\breve{a}\breve{b}+\breve{b}\breve{c}+\breve{c}\breve{a}-1} = 0.$
And it is clear that $\breve{a}\breve{b}\breve{c} - \breve{a} - \breve{b} - \breve{c} = 0 \Leftrightarrow \breve{a}\breve{b}\breve{c} = \breve{a} + \breve{b} + \breve{c}$
Remark. The denominator cannot be zero. In an obtuse or right triangle we have $0 < \breve{a} \leq 1$ and $\breve{b}, \breve{c} > 1$ and in an acute triangle all three are greater than one.

1.2 The functions $c(x) = \frac{x^2-1}{x^2+1}, s(x) = \frac{2x}{x^2+1}$

These functions are related to the cosine and the sine as functions of the cotangents of half of the arguments.

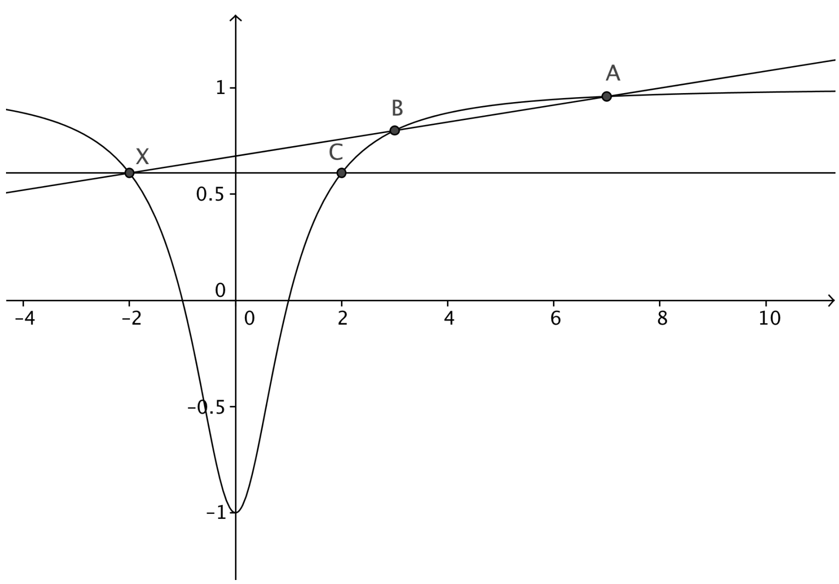

Figure 1: graphic of $c(x) = \frac{x^2-1}{x^2+1}$

Let $A(\breve{a}, c(\breve{a}))$ and $B(\breve{b}, c(\breve{b}))$ be two different points on the graph of c. Then the slope of the line AB is $\frac{c(\breve{a})-c(\breve{b})}{\breve{a}-\breve{b}} = \frac{2(\breve{a}+\breve{b})}{(\breve{a}^2+1)(\breve{b}^2+1)}$ and the equation of this line is

$$y - \frac{\breve{a}^2-1}{\breve{a}^2+1} = \frac{2(\breve{a}+\breve{b})}{(\breve{a}^2+1)(\breve{b}^2+1)}(x-\breve{a}) \quad (1)$$

or equivalently

$$y = \frac{2(\breve{a}+\breve{b})}{(\breve{a}^2+1)(\breve{b}^2+1)} \cdot x + \frac{\breve{a}^2\breve{b}^2 - 1 - (\breve{a}+\breve{b})^2}{(\breve{a}^2+1)(\breve{b}^2+1)} \tag{2}$$

This combining with the equation of the graph of $c(x)$

$$y = \frac{x^2-1}{x^2+1} \tag{3}$$

and setting $y = px + q$ for the line AB gives

$$px + q = \frac{x^2-1}{x^2+1} \Leftrightarrow px^3 + (q-1)x^2 + px + (q+1) = 0 \tag{4}$$

The solutions of this cubic equation are $x_1 = \breve{a}$, $x_2 = \breve{b}$ and $x_3 = -\frac{q+1}{\breve{a}\breve{b}p} = -\frac{\breve{a}\breve{b}-1}{\breve{a}+\breve{b}}$. The third common point of the line AB and the graph of $c(x)$ is $X = \left(-\frac{\breve{a}\breve{b}-1}{\breve{a}+\breve{b}}, \frac{(\breve{a}\breve{b}-1)^2-(\breve{a}+\breve{b})^2}{(\breve{a}^2+1)(\breve{b}^2+1)}\right)$. Connecting this point X with the point at infinity in the direction of the x-axis gives on the graph of $c(x)$ the point $C = \left(\frac{\breve{a}\breve{b}-1}{\breve{a}+\breve{b}}, \frac{(\breve{a}\breve{b}-1)^2-(\breve{a}+\breve{b})^2}{(\breve{a}^2+1)(\breve{b}^2+1)}\right) = (\breve{c}(\alpha+\beta), c(\breve{c})) = \left(\cot\frac{1}{2}(\alpha+\beta), \cos(\alpha+\beta)\right)$
From the preceeding follows:

Lemma.
Let $\mathbb{B}$ be the set of rational points of the graph of $c(x) = \frac{x^2-1}{x^2+1}$ with addition of the point at infinity. Then $\mathbb{B}$ with the operation described above is a group with unit-element the point at infinity.

The graph of $s(x) = \frac{2x}{x^2+1}$ has similar properties. See figure.:
The line through $P = (\breve{a}, \frac{2\breve{a}}{\breve{a}^2+1})$ and $Q = (\breve{b}, \frac{2\breve{b}}{\breve{b}^2+1})$ intersects the graph of c in a third point $Y = (\frac{\breve{a}+\breve{b}}{\breve{a}\breve{b}-1}, \frac{2(\breve{a}+\breve{b})(\breve{a}\breve{b}-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}) = Y(\frac{\breve{a}+\breve{b}}{\breve{a}\breve{b}-1}, \sin(\alpha+\beta))$. And again connecting this point Y with the point at infinity we get $R = (\frac{\breve{a}\breve{b}-1}{\breve{a}+\breve{b}}, \sin(\alpha+\beta))$

Lemma.
Let $(\mathbb{Q}^{\oplus}, \oplus)$ be the set of rational numbers with addition of the element ∞. Let the operation $\oplus$ be defined by:

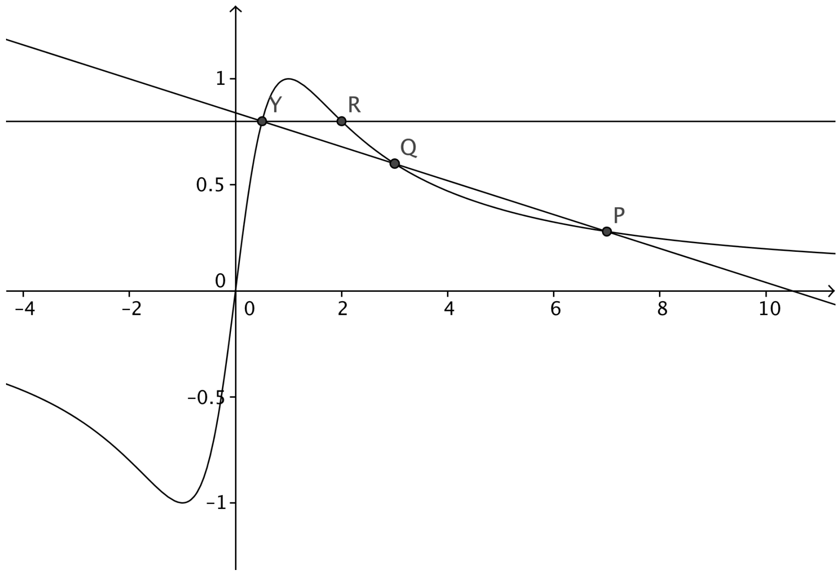

Figure 2: graphic of $s(x) = \frac{2x}{x^2+1}$

$$a \oplus b = \begin{cases} \frac{ab-1}{a+b} & \text{if} \quad a, b \in \mathbb{Q} \\ \frac{b-\frac{1}{a}}{1+\frac{b}{a}} = b & \text{if} \quad a = \infty \text{ and } b \in \mathbb{Q} \\ \frac{1-\frac{1}{ab}}{\frac{1}{a}+\frac{1}{b}} = \frac{1}{0} = \infty & \text{if} \quad a = b = \infty \end{cases}$$

Then $(\mathbb{Q}^{\oplus}, \oplus)$ is an abelian group.
Proof.
∞ is unit-element.
$a \oplus b = \frac{ab-1}{ab+b} = \infty \Leftrightarrow a = -b$
$(a \oplus b) \oplus c = \frac{\frac{ab-1}{a+b} \cdot c - 1}{\frac{ab-1}{a+b} + c} = \frac{abc-a-b-c}{ab+bc+ca-1}$ $a \oplus (b \oplus c) = \frac{abc-a-b-c}{ab+bc+ca-1}$.

From the preceeding now follows easily the
Proposition.
Let $\mathbb{A}$ be the group of Q-angles with usual addition modulo 2π or $360°$. Let $\mathbb{B}$ be the group of rational points on the graph of $c(x) = \frac{x^2-1}{x^2+1}$ with the point at infinity as neutral element as above. Then there are isomorphisms between $\mathbb{A}$, $\mathbb{Q}^{\oplus}$ and $\mathbb{B}$. These isomorphisms identify the usual addition of angles with the groupoperation on the curve $(x^2+1)y = x^2 - 1$.

2 Q-triangles given by three rationals.

In this section we discuss some problems for the construction of a Q-triangle, given three rationals. These rationals may be the cotangens of half an angle of the triangle, or be the length of a side of the triangle.

2.1 AAA

By the proposition at the end of section 1.1 $\breve{a}$, $\breve{b}$ and $\breve{c}$ must satisfy the condition $\breve{a} \cdot \breve{b} \cdot \breve{c} = \breve{a} + \breve{b} + \breve{c}$. The lenght of a side and the area of the triangle remain unknown.

2.2 ASA

Exercise.
Let be given $\breve{a}, \breve{b}, c \in \mathbb{Q}$.
Show:
$\breve{c} = \frac{\breve{a}+\breve{b}}{\breve{a}\breve{b}-1}$, $a = \frac{\breve{a}(\breve{c}^2+1)}{\breve{c}(\breve{a}^2+1)}c$ and $b = \frac{\breve{b}(\breve{c}^2+1)}{\breve{c}(\breve{b}^2+1)}c$
The construction of Q-triangle ABC is always possible.
Example.
$\breve{a} = \frac{3}{2}, \breve{b} = 2$ and $c = 14$ gives $\breve{c} = \frac{7}{4}, a = 15$ and $b = 13$

2.3 AAS

Exercise.
Let be given $\breve{a}, \breve{b}, a \in \mathbb{Q}$.
Show:
$\breve{c} = \frac{\breve{a}+\breve{b}}{\breve{a}\breve{b}-1}$, $c = \frac{\breve{c}(\breve{a}^2+1)}{\breve{a}(\breve{c}^2+1)}a$ and $b = \frac{\breve{b}(\breve{c}^2+1)}{\breve{c}(\breve{b}^2+1)}c$
The construction of Q-triangle ABC is always possible.
Example.
$\breve{a} = \frac{3}{2}, \breve{b} = 2$ and $a = 15$ gives $\breve{c} = \frac{7}{4}, c = 14$ and $b = 13$

2.4 SAS

Let be given $a, b, \check{c} \in \mathbb{Q}$ and let $p = \frac{b}{a}$
Then
$p = \frac{b}{a} = \frac{\sin\beta}{\sin\alpha} \Leftrightarrow \frac{1+p}{1-p} = \frac{\sin(\alpha)+\sin(\beta)}{\sin(\alpha)-\sin(\beta)} = \frac{\sin\frac{1}{2}(\alpha+\beta)\cos(\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha-\beta)\cos(\frac{1}{2}(\alpha+\beta)} = \frac{\cot\frac{1}{2}(\alpha-\beta)}{\cot\frac{1}{2}(\alpha+\beta)}$
$\cot\frac{1}{2}(\alpha-\beta) = \frac{\check{a}\check{b}+1}{\check{b}-\check{a}}$
In $\triangle ABC$ is $\cot\frac{1}{2}(\alpha+\beta) = \tan(90^\circ - \frac{1}{2}(\alpha+\beta)) = \tan\frac{1}{2}\gamma = \frac{1}{\cot\frac{1}{2}\gamma} = \frac{1}{\check{c}}$
Combination of the last three equations gives:
$\frac{1+p}{1-p} = \frac{\cot\frac{1}{2}(\alpha-\beta)}{\cot\frac{1}{2}(\alpha+\beta)} = \check{c}\frac{\check{a}\check{b}+1}{\check{b}-\check{a}} \Rightarrow (1+p)(\check{b}-\check{a}) = (1-p)\check{c}(\check{a}\check{b}+1)$
Substitution of $\check{a} = \frac{\check{b}+\check{c}}{\check{b}\check{c}-1}$ gives:

$$(1+p)(\check{b} - \frac{\check{b}+\check{c}}{\check{b}\check{c}-1}) = (1-p)\check{c}(\frac{\check{b}+\check{c}}{\check{b}\check{c}-1}\check{b}+1) \tag{5}$$

$$\Leftrightarrow \check{c}p\cdot\check{b}^2 + (\check{c}^2p - \check{c}^2 - p - 1)\check{b} - \check{c}p = 0 \tag{6}$$

or

$$\check{b}^2 - \left(\frac{\check{c}^2+1}{\check{c}p} - \frac{\check{c}^2-1}{\check{c}}\right)\check{b} - 1 = 0 \tag{7}$$

The discriminant of this last equation is $D = \left(\frac{\check{c}^2+1}{\check{c}p} - \frac{\check{c}^2-1}{\check{c}}\right)^2 + 4$ and after some computation :

$$D = \left(\frac{\check{c}^2+1}{\check{c}p}\right)^2\left(p^2 - 2p\frac{\check{c}^2-1}{\check{c}^2+1} + 1\right) \tag{8}$$

$$D = \left(\frac{\check{c}^2+1}{\check{c}p}\right)^2 \cdot A \text{ with } A = \left((p-\cos\gamma)^2 + \sin^2\gamma\right) \tag{9}$$

Conclusion.
There is a Q-solution for this SAS-problem, when $A = (p-\cos\gamma)^2 + \sin^2\gamma$ is a square of a rational.

Thus given $a, b, \check{c} \in \mathbb{Q}$ and let $p = \frac{b}{a}$ with $\left((p-\cos\gamma)^2 + \sin^2\gamma\right)$ is a square of a rational gives:

$$\check{b} = \frac{1}{2}\left(\frac{\check{c}^2+1}{\check{c}p} - \frac{\check{c}^2-1}{\check{c}} \pm \left(\frac{\check{c}^2+1}{\check{c}p}\right)\sqrt{A}\right) \tag{10}$$

or

$$\check{b} = \frac{1 - p\cos\gamma \pm \sqrt{(p-\cos\gamma)^2 + \sin^2\gamma}}{p\sin\gamma} \tag{11}$$

$$\check{a} = \frac{\check{b} + \check{c}}{\check{b}\check{c} - 1} \tag{12}$$

Example.
$\check{c} = \frac{7}{4}$ and $b = 13$ and $a = 15$.
Then $p = \frac{13}{15}$ and
$A = \left(\frac{13}{15} - \frac{33}{65}\right)^2 + \left(\frac{56}{65}\right)^2 = \left(\frac{14}{39}\right)^2 + \left(\frac{56}{65}\right)^2 = \left(\frac{14}{13}\right)^2 \left(\left(\frac{1}{3}\right)^2 + \left(\frac{4}{5}\right)^2\right) = \left(\frac{14}{13}\right)^2 \left(\frac{13}{15}\right)^2$
$\check{b} = \frac{1 - \frac{13}{15}\frac{33}{65} \pm \frac{14}{13}\frac{13}{15}}{\frac{13}{15}\frac{56}{65}} = \frac{1 - \frac{33}{75} \pm \frac{14}{15}}{\frac{56}{75}} = \frac{75 - 33 \pm 70}{56} \Rightarrow \check{b} = 2 \vee \check{b} = -\frac{1}{2}$
$\check{b} > 0 \Rightarrow \check{b} = 2$
$\check{a} = \frac{2 + \frac{7}{4}}{2 \cdot \frac{7}{4} - 1} = \frac{3}{2}$
$c = \frac{\sin\gamma}{\sin\beta} b = \frac{56}{65} \cdot \frac{5}{4} \cdot 13 = 14$

At the end of this subsection we generate values of p and in consequence constructions of Q-triangles with given $\check{c}$.
Let $\angle ACD = \gamma \smile \check{c}$ and $AC = b$ and $AD \perp BC$. Let $\angle(BAD) = \xi \smile \check{x}$.

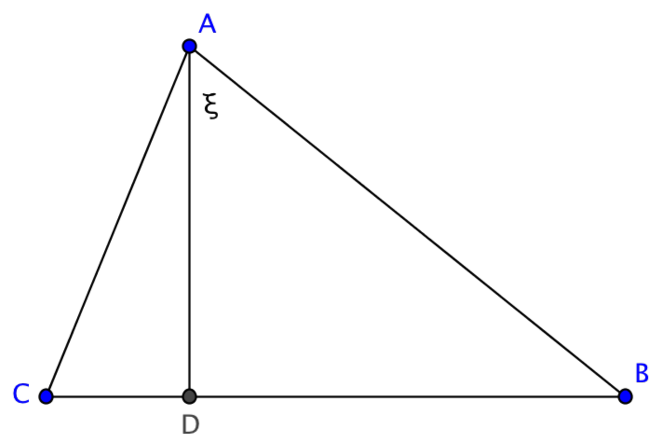

Figure 3: $\angle\xi$

Then $BC = CD + BD = b\cos\gamma + AD\tan\xi = b\cos\gamma + b\sin\gamma\tan\xi$

$p = \frac{AC}{BC} = \frac{\sin\beta}{\sin(90^\circ - \gamma + \xi)} = \frac{\cos\xi}{\cos(\gamma - \xi)} = \frac{(\breve{c}^2+1)(\breve{x}^2-1)}{(\breve{c}^2-1)(\breve{x}^2-1)+4\breve{c}\breve{x}}$
And finally

$$p = \frac{\breve{x}^2 - 1}{(\breve{x}^2 - 1)\cos\gamma + 2\breve{x}\sin\gamma} \tag{13}$$

Thie completes the proof of :

Proposition.
Let $\gamma \smile \breve{c}$ be any Q-angle. Then for every $\breve{x} \in \mathbb{Q}$ there exists a ratio $p = \frac{b}{a}$.

The inverse is not valid.
For example: Let $p \in Q$. Then
$p = \frac{(\breve{c}^2+1)(\breve{x}^2-1)}{(\breve{c}^2-1)(\breve{x}^2-1)+4\breve{c}\breve{x}} \Leftrightarrow (\breve{c}^2 + 1 - p(\breve{c}^2 - 1)\breve{x}^2 - 4p\breve{c}\breve{x} - (\breve{c}^2 + 1 - p(\breve{c}^2 - 1) = 0$
$\breve{x}^2 - 2p\frac{\sin\gamma}{1-p\cos\gamma}\breve{x} - 1 = 0$ with discriminant $D = 4p^2\frac{\sin^2\gamma}{(1-p\cos\gamma)^2} + 4$ and $\sqrt{D}$ is not always a number in $\mathbb{Q}$ as you can see by $p = \frac{3}{2}$ and $\breve{c} = 2$.

Examples:
$\breve{x} = 1 \Rightarrow \xi = 90 \Rightarrow AB//CD$
$\breve{x} = \breve{c} \Rightarrow p = \frac{\breve{c}^2-1}{(\breve{c}^2-1)\cos\gamma+2\breve{c}\sin\gamma} = \frac{\cos\gamma}{\cos^2\gamma+\sin^2\gamma} = \cos\gamma$

2.5 ASS

Let be given $\breve{a}, a, c, \in \mathbb{Q}$ see fig.
Let $BD \perp AC$ and let $\angle(CBD) = \xi \smile \breve{x}$.
Then $\cos\xi = \frac{\breve{x}^2-1}{\breve{x}^2+1} = \frac{c\sin\alpha}{a} = \frac{2\breve{a}c}{(\breve{a}^2+1)a}$ and $a(\breve{a}^2+1)\breve{x}^2 - a(\breve{a}^2+1) = 2c\breve{a}\breve{x}^2 + 2c\breve{a}$
And finally:

$$\breve{x}^2 = \frac{a(\breve{a}^2 + 1) + 2c\breve{a}}{a(\breve{a}^2 + 1) - 2c\breve{a}} = \frac{a + c\sin\alpha}{a - c\sin\alpha} \tag{14}$$

We see that with the given $\breve{a}, a, c, \in \mathbb{Q}$ a Q-triangle is possible if $\frac{a+c\sin\alpha}{a-c\sin\alpha}$ is a square of a rational number.
As in the preceeding subsection every $x \in \mathbb{Q}$ generates with given $a, \breve{a} \in \mathbb{Q}$ a natural number c such that the Q-triangle given by $\breve{a}, a, c, \in \mathbb{Q}$ exists.

$$c = \frac{a(\breve{a}^2 + 1)(\breve{x}^2 - 1)}{2\breve{a}(\breve{x}^2 + 1)} = \frac{a}{\sin\alpha} \cdot \frac{\breve{x}^2 - 1}{\breve{x}^2 + 1} \tag{15}$$

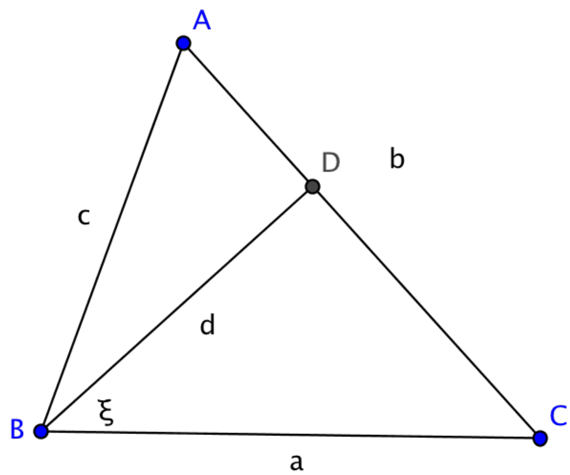

Figure 4: SSA

Example.
$\breve{a} = \frac{3}{2}, a = 15, c = 14.$
Solution. $\breve{x}^2 = \frac{15+14\cdot\frac{12}{13}}{15-14\cdot\frac{12}{13}} = \frac{13\cdot15+14\cdot12}{13\cdot15+14\cdot12} = \frac{363}{27} = \left(\frac{11}{3}\right)^2 \Leftrightarrow \breve{x} = -\frac{11}{3} \vee \breve{x} = \frac{11}{3}$
$\sin\xi = -\frac{33}{65} \vee \sin\xi = \frac{33}{65}$
$b = c\cos\alpha + a\sin\xi \Rightarrow b = 14\cdot\frac{5}{13} \pm 15\cdot\frac{33}{65} = \frac{70}{13} \pm \frac{99}{13} = \frac{169}{13} = 13$. The minus sine is not valid, because C lies on the half line AD.
The angles β and γ now follow easily, using the sine-rule.

2.6 SSS

Let be given $a, b, c \in \mathbb{Q}$.
Let $s = \frac{a+b+c}{2}$. Then $\triangle(ABC)$ is an Q-triangle if and only if $\sqrt{s(s-a)(s-b)(s-c)} \in (Q)$. The angles follow easily, using the cosine-rule.

3 Two examples of Q-triangle, divided in Q-triangles by cevians through an inner point and/or the pedals of that inner point on the sides.

3.1 6 triangles around the orthocenter.

All Q-triangles are devided into two Pythagorean triangles by each of their altitudes. And thus the three altitudes devide the triangle in 6 small Pythagorean triangles. So we have a Q-configuration consisting of 12 side-lenghts and 6 triangles. In the next table you can find the $\breve{a}$, $\breve{b}$ and $\breve{c}$ and the lengths of the 12 linesegments and the 6 areas of the small triangles surrounding the orthocenter H of the Q-triangle ABC. Additional are the lengths of the sides and altitudes and area of other triangles. In the first two columns you will find left the well-known triangle 13-14-15 by Heron and at right blown up to whole numbers. In the following two columns you will find another smaller Q-triangle. In the last column the same lengths and areas rearranged, when the orthocenter lies outside the triangle. See fig 2.

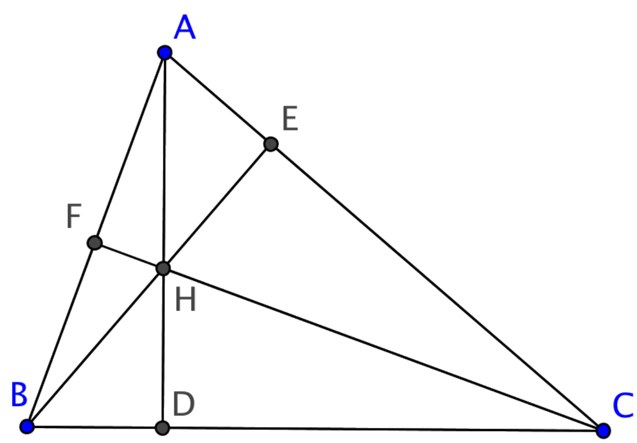

Figure 5: 6 triangles around orthocenter

Let $\angle(\alpha) \smile \breve{a}$, $\angle(\beta) \smile \breve{b}$ and the radius of the circumcirkle R. The formulas for the sides, expressed in $\breve{a}, \breve{b}, \text{and} R$, are as follows:

$\breve{c} = \frac{\breve{a}+\breve{b}}{\breve{a}\breve{b}-1}$

$\sin\alpha = \frac{2\breve{a}}{\breve{a}^2+1}$ and $\cos\alpha = \frac{\breve{a}^2-1}{\breve{a}^2+1}$

$\sin\beta = \frac{2\breve{b}}{\breve{b}^2+1}$ and $\cos\beta = \frac{\breve{b}^2-1}{\breve{b}^2+1}$

$\sin\gamma = \frac{2\breve{c}}{\breve{c}^2+1} = \frac{2\frac{\breve{a}+\breve{b}}{\breve{a}\breve{b}-1}}{(\frac{\breve{a}+\breve{b}}{\breve{a}\breve{b}-1})^2+1} = \frac{2(\breve{a}+\breve{b})(\breve{a}\breve{b}-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}$

$\cos\gamma = \frac{\breve{c}^2-1}{\breve{c}^2+1} = \frac{(\frac{\breve{a}+\breve{b}}{\breve{a}\breve{b}-1})^2-1}{(\frac{\breve{a}+\breve{b}}{\breve{a}\breve{b}-1})^2+1} = \frac{4\breve{a}\breve{b}-(\breve{a}^2-1)(\breve{b}^2-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}$

$a = \frac{4\breve{a}}{\breve{a}^2+1}R$

$b = \frac{4\breve{b}}{\breve{b}^2+1}R$

$c = \frac{4(\breve{a}+\breve{b})(\breve{a}\breve{b}-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}R$

$AE = c\cos\alpha = \frac{\breve{a}^2-1}{\breve{a}^2+1}\frac{4(\breve{a}+\breve{b})(\breve{a}\breve{b}-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}R = \frac{4(\breve{a}^2-1)(\breve{a}+\breve{b})(\breve{a}\breve{b}-1)}{(\breve{a}^2+1)^2(\breve{b}^2+1)}R$

$AF = b\cos\alpha = \frac{4\breve{b}}{\breve{b}^2+1}\frac{\breve{a}^2-1}{\breve{a}^2+1}R = \frac{4\breve{b}(\breve{a}^2-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}R$

$BF = a\cos\beta = \frac{4\breve{a}}{\breve{a}^2+1}\frac{\breve{b}^2-1}{\breve{b}^2+1}R = \frac{4\breve{a}(\breve{b}^2-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}R$

$BD = c\cos\beta = \frac{\breve{b}^2-1}{\breve{b}^2+1}\frac{4(\breve{a}+\breve{b})(\breve{a}\breve{b}-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}R = \frac{4(\breve{b}^2-1)(\breve{a}+\breve{b})(\breve{a}\breve{b}-1)}{(\breve{a}^2+1)(\breve{b}^2+1)^2}R$

$CD = b\cos\gamma = \frac{4\breve{b}}{\breve{b}^2+1}\frac{4\breve{a}\breve{b}-(\breve{a}^2-1)(\breve{b}^2-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}R = \frac{4\breve{b}(4\breve{a}\breve{b}-(\breve{a}^2-1)(\breve{b}^2-1))}{(\breve{a}^2+1)(\breve{b}^2+1)^2}R$

$CE = a\cos\gamma = \frac{4\breve{a}}{\breve{a}^2+1}\frac{4\breve{a}\breve{b}-(\breve{a}^2-1)(\breve{b}^2-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}R = \frac{4\breve{a}(4\breve{a}\breve{b}-(\breve{a}^2-1)(\breve{b}^2-1))}{(\breve{a}^2+1)^2(\breve{b}^2+1)}R$

$HA = \frac{AF}{\sin\beta} = \frac{4\breve{b}(\breve{a}^2-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}\frac{\breve{b}^2+1}{2\breve{b}}R = \frac{2(\breve{a}^2-1)}{\breve{a}^2+1}R = 2R\cos\alpha$

$HB = \frac{BF}{\sin\alpha} = \frac{4\breve{a}(\breve{b}^2-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}\frac{\breve{a}^2+1}{2\breve{a}}R = \frac{2(\breve{b}^2-1)}{\breve{b}^2+1}R = 2R\cos\beta$

$HC = \frac{CD}{\sin\beta} = \frac{4\breve{b}(4\breve{a}\breve{b}-(\breve{a}^2-1)(\breve{b}^2-1))}{(\breve{a}^2+1)(\breve{b}^2+1)^2}\frac{\breve{b}^2+1}{2\breve{b}}R = \frac{2(4\breve{a}\breve{b}-(\breve{a}^2-1)(\breve{b}^2-1))}{(\breve{a}^2+1)(\breve{b}^2+1)}R = 2R\cos\gamma$

$HD = HB\cos\gamma = \frac{2(\breve{b}^2-1)}{\breve{b}^2+1}\frac{4\breve{a}\breve{b}-(\breve{a}^2-1)(\breve{b}^2-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}R = \frac{2(\breve{b}^2-1)(4\breve{a}\breve{b}-(\breve{a}^2-1)(\breve{b}^2-1))}{(\breve{a}^2+1)(\breve{b}^2+1)^2}R$

$HE = HC\cos\alpha = \frac{2((\breve{a}+\breve{b})^2-(\breve{a}\breve{b}-1)^2)}{(\breve{a}^2+1)(\breve{b}^2+1)}\frac{\breve{a}^2-1}{\breve{a}^2+1}R = \frac{2((\breve{a}+\breve{b})^2-(\breve{a}\breve{b}-1)^2)(\breve{a}^2-1)}{(\breve{a}^2+1)^2(\breve{b}^2+1)}R$

$HF = HA\cos\beta = \frac{2(\breve{a}^2-1)(\breve{b}^2-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}R = 2R\cos\alpha\cos\beta$

$\breve{a}$		7/4		13/9	11/2
$\breve{b}$		3/2		4/3	7
$\breve{c}$		2		3	1/3
AC	(15)	(3900)	(40)	(3000)	875
BC	(14)	(3640)	(39)	(2925)	1100
AB	(13)	(3380)	(25)	(1875)	(1875)
ABC	(84)	(5678400)	(468)	(2632500)	(288750)
HA	33/4	2145	44/3	1100	(2925)
HB	25/4	1625	35/3	875	(3000)
HC	39/4	2535	100/3	2500	2500
HD	15/4	975	28/3	700	2400
HE	99/20	1287	176/15	880	2340
HF	165/52	825	308/75	308	(2808)
AE	33/5	1716	44/5	660	(1755)
EC	42/5	2184	156/5	2340	880
CD	9	2340	32 273/25	2400	700
DB	5	1300	7	525	(1800)
BF	70/13	1400	273/25	819	1056
FA	99/13	1980	352/25	1056	819
HAE	3267/200	1104246	3872/75	290400	(1149876)
HEC	2079/100	1405404	4576/25	1029600	1029600
HCD	135/8	1140750	448/3	840000	840000
HDB	75/8	633750	98/3	183750	(2160000)
HBF	5775/676	577500	14014/625	126126	(1482624)
HFA	16335/1352	816750	54208/1875	162624	(2053350
AD	(12)	(3120)	(24)	(1800)	525
BE	(56/5)	(2912)	(117/5)	(1755)	660
CF	(168/13)	(3360)	(936/25)	(2808)	308
BEC	(1176/25)	(3179904)	(9126/25)	(2053350)	290400
BCF	(5880/169)	(2352000)	(127764/625)	(1149876)	162624
BCH	(105/4)	(1774500)	(182)	(1023750)	(1029600)
ABE	(924/25)	(2498496)	(2574/25)	(579150)	(579150)
ABD	(30)	(2028000)	(84)	(472500)	(472500)
ABH	(165/8)	(1394250)	(154/3)	(288750)	(2632500)
ACF	(8316/169)	(3326400)	(164736/625)	(1482624)	(126126)
ADC	(54)	(3650400)	(384)	(2160000)	(183750)
ACH	(297/8)	(2509650)	(704/3)	(1320000)	(1023750)

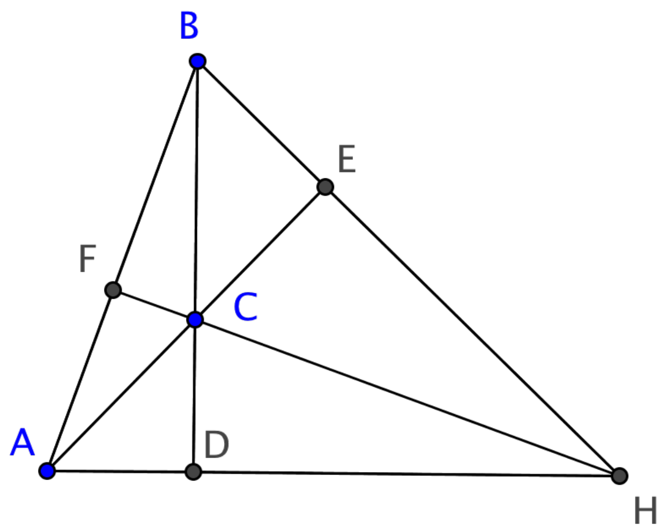

Figure 6: 6 triangles around orthocenter with orthocenter outside

3.2 9 triangles around the circumcenter

Here you will find the formulas concerning a triangle with circumcenter O. The cevians AO, BO and CO meet the opposite sides in P, Q and R respectively. The midpoints of BC, AC and AB are D, E and F respectively. A well-known property of this configuration is $\angle AOB = 2\angle ACB$. Using this property we will express the lenghts of the 18 line segments in $\breve{a}, \breve{b}$ and R.

$\breve{c} = \frac{\breve{a}+\breve{b}}{\breve{a}\breve{b}-1}$

$\sin\alpha = \frac{2\breve{a}}{\breve{a}^2+1}$ and $\cos\alpha = \frac{\breve{a}^2-1}{\breve{a}^2+1}$

$\sin\beta = \frac{2\breve{b}}{\breve{b}^2+1}$ and $\cos\beta = \frac{\breve{b}^2-1}{\breve{b}^2+1}$

$\sin\gamma = \frac{2\breve{c}}{\breve{c}^2+1} = \frac{2\frac{\breve{a}+\breve{b}}{\breve{a}\breve{b}-1}}{(\frac{\breve{a}+\breve{b}}{\breve{a}\breve{b}-1})^2+1} = \frac{2(\breve{a}+\breve{b})(\breve{a}\breve{b}-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}$

$\cos\gamma = \frac{\breve{c}^2-1}{\breve{c}^2+1} = \frac{(\frac{\breve{a}+\breve{b}}{\breve{a}\breve{b}-1})^2-1}{(\frac{\breve{a}+\breve{b}}{\breve{a}\breve{b}-1})^2+1} = \frac{4\breve{a}\breve{b}-(\breve{a}^2-1)(\breve{b}^2-1)}{(\breve{a}^2+1)(\breve{b}^2+1)}$

$\cos(\xi-\eta) = \frac{(\breve{x}^2-1)(\breve{y}^2-1)+4\breve{x}\breve{y}}{(\breve{x}^2+1)(\breve{y}^2+1)}$

$BD = R\sin\alpha = \frac{2\breve{a}}{\breve{a}^2+1}R$

$AE = R\sin\beta = \frac{2\breve{b}}{\breve{b}^2+1}R$

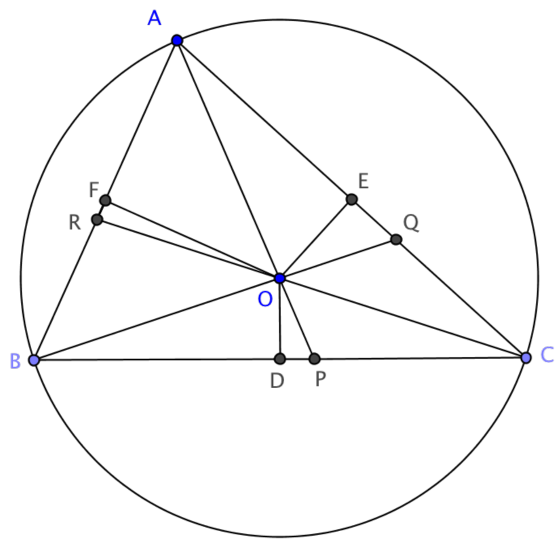

Figure 7: 9 triangles around circumcenter

$AF = R\sin\gamma = \frac{2(\check{a}+\check{b})(\check{a}\check{b}-1)}{(\check{a}^2+1)(\check{b}^2+1)}R$

$OD = R\cos\alpha = \frac{\check{a}^2-1}{\check{a}^2+1}R$

$OE = R\cos\beta = \frac{\check{b}^2-1}{\check{b}^2+1}R$

$OF = R\cos\gamma = \frac{4\check{a}\check{b}-(\check{a}^2-1)(\check{b}^2-1)}{(\check{a}^2+1)(\check{b}^2+1)}R$

$OP = \frac{\sin\angle OBP}{\sin\angle P}R = \frac{\cos\alpha}{cos(\beta-\gamma)}R = \frac{(\check{a}^2-1)(\check{b}^2+1)(\check{c}^2+1)}{(\check{a}^2+1)((\check{b}^2-1)(\check{c}^2-1)+4\check{b}\check{c})}R$

$OQ = \frac{(\check{b}^2-1)(\check{c}^2+1)(\check{a}^2+1)}{(\check{b}^2+1)((\check{c}^2-1)(\check{a}^2-1)+4\check{c}\check{a})}R$

$OR = \frac{(\check{c}^2-1)(\check{a}^2+1)(\check{b}^2+1)}{(\check{c}^2+1)((\check{a}^2-1)(\check{b}^2-1)+4\check{a}\check{b})}R$

$DP = OP\sin(\beta-\gamma) = \cos\alpha\tan(\beta-\gamma)R = \frac{(\check{a}^2-1)(2\check{b}(\check{c}^2-1)-2\check{c}(\check{b}^2-1)}{(\check{a}^2+1)(\check{b}^2+1)(\check{c}^2+1)}R$

$EQ = \frac{(\check{b}^2-1)(2\check{a}(\check{c}^2-1)-2\check{c}(\check{a}^2-1)}{(\check{a}^2+1)(\check{b}^2+1)(\check{c}^2+1)}R$

$FR = \frac{(\check{c}^2-1)(2\check{a}(\check{b}^2-1)-2\check{b}(\check{a}^2-1)}{(\check{a}^2+1)(\check{b}^2+1)(\check{c}^2+1)}R$

$PC = CD - DP = \frac{2\check{a}(\check{b}^2+1)(\check{c}^2+1)-(\check{a}^2-1)(2\check{b}(\check{c}^2-1)-2\check{c}(\check{b}^2-1)}{(\check{a}^2+1)(\check{b}^2+1)(\check{c}^2+1)}R$

$QC = CE - EQ = \frac{2\check{b}(\check{a}^2+1)(\check{c}^2+1)-(\check{b}^2-1)(2\check{a}(\check{c}^2-1)-2\check{c}(\check{a}^2-1)}{(\check{a}^2+1)(\check{b}^2+1)(\check{c}^2+1)}R$

$RB = BF - FR = \frac{2\check{c}(\check{a}^2+1)(\check{b}^2+1)-(\check{c}^2-1)(2\check{a}(\check{b}^2-1)-2\check{b}(\check{a}^2-1)}{(\check{a}^2+1)(\check{b}^2+1)(\check{c}^2+1)}R$

Example.
$\breve{a} = 3$, $\breve{b} = \frac{4}{3}$ and $R = 125$.

OA	125	$5^3 \cdot 17 \cdot 19 \cdot 31 \cdot 41$	51316625
OR	$100\frac{225}{779}$	$5^7 \cdot 17 \cdot 31$	41171875
OF	100	$2^2 \cdot 5^2 \cdot 17 \cdot 19 \cdot 31 \cdot 41$	41053300
OB	125	$5^3 \cdot 17 \cdot 19 \cdot 31 \cdot 41$	51316625
OP	55	$5 \cdot 11 \cdot 17 \cdot 19 \cdot 31 \cdot 41$	22579315
OD	44	$2^2 \cdot 11 \cdot 17 \cdot 19 \cdot 31 \cdot 41$	18063452
OC	125	$5^3 \cdot 17 \cdot 19 \cdot 31 \cdot 41$	51316625
OQ	$41\frac{268}{527}$	$5^5 \cdot 7 \cdot 19 \cdot 41$	17040625
OE	35	$5 \cdot 7 \cdot 17 \cdot 19 \cdot 31 \cdot 41$	14368655
AR	$67\frac{307}{779}$	$2^2 \cdot 3 \cdot 5^4 \cdot 7 \cdot 17 \cdot 31$	27667500
RF	$7\frac{472}{779}$	$3 \cdot 5^2 \cdot 17 \cdot 31 \cdot 79$	3122475
FB	75	$3 \cdot 5^2 \cdot 17 \cdot 19 \cdot 31 \cdot 41$	30789975
BD	117	$3^2 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 41$	48032361
DP	33	$3 \cdot 11 \cdot 17 \cdot 19 \cdot 31 \cdot 41$	13547589
PC	84	$2^2 \cdot 3 \cdot 7 \cdot 17 \cdot 19 \cdot 31 \cdot 41$	34484772
CQ	$97\frac{361}{527}$	$2^3 \cdot 3^2 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \cdot 41$	40102920
QE	$22\frac{166}{527}$	$2^4 \cdot 3 \cdot 5 \cdot 7^2 \cdot 19 \cdot 41$	9161040
EA	120	$2^3 \cdot 3 \cdot 5 \cdot 17 \cdot 19 \cdot 31 \cdot 41$	49263960
OAR	$\frac{2625000}{779}$	$2^3 \cdot 3 \cdot 5^6 \cdot 7 \cdot 17^2 \cdot 19 \cdot 31^2 \cdot 41$	567921088875000
ORF	$\frac{296250}{779}$	$2 \cdot 3 \cdot 5^4 \cdot 17^2 \cdot 19 \cdot 31^2 \cdot 41 \cdot 79$	64093951458750
OFB	3750	$2 \cdot 5^4 \cdot 17^2 \cdot 19^2 \cdot 31^2 \cdot 41^2$	632015040333750
OBD	2574	$2 \cdot 3^2 \cdot 11 \cdot 13 \cdot 17^2 \cdot 19^2 \cdot 31^2 \cdot 41^2$	433815123685086
ODP	726	$2 \cdot 3 \cdot 11^2 \cdot 17^2 \cdot 19^2 \cdot 31^2 \cdot 41^2$	122358111808614
OPC	1848	$2^3 \cdot 3 \cdot 7 \cdot 11 \cdot 17^2 \cdot 19^2 \cdot 31^2 \cdot 41^2$	311457011876472
OCQ	$\frac{900900}{527}$	$2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19^2 \cdot 31 \cdot 41^2$	288112510986300
OQE	$\frac{205800}{527}$	$2^3 \cdot 3 \cdot 5^2 \cdot 7^3 \cdot 17 \cdot 19^2 \cdot 31 \cdot 41^2$	65815911600600
OEA	2100	$2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 17^2 \cdot 19^2 \cdot 31^2 \cdot 41^2$	353928422586900
ABC			2839517173211472