Q-quadrangles inscribed in a circle

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Abstract

A brief introduction with some examples of Q-configurations. In the second part some properties about the length of the sides of Q-quadrangles, inscribed in a circle. At the end some remarks on almost regular Q-polygons.

Contents

1 Introduction

Pythagoras and later Heron were occupied with (right) triangles with sidelenghts and area measured in whole numbers. There are much properties and theorems about these triangles. Cf 'On Triangles with rational altitudes, angle bisectors of medians by RH Buchholz (1989) You can find the following formulas in many texts:

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 - n^2 \tag{1}$$

Then some remarks on requirements for m en n to get a so called primitive triangle. Primitive means that sidelengths and area are whole numbers with $GGD(a, b, c, area) = 1$ is.
An other approach is to begin with the angles of the triangle. All triangles with the same angles are similar. The angles of a Pythagorean triangle are determined by one complex number from the squares of the whole complex numbers. There exists a 1-1 correspondence between these angles and the the ideals of the squares of the whole complex numbers. And the ideal can be represented by a point on the circle with radius 1. Thes angles are called Q-angles and are denoted by $\angle A = \cos(\alpha) + \sin(\alpha).\imath$, with

$$\sin(\alpha) = \frac{2a_1a_0}{a_1^2 + a_0^2} \qquad \cos(\alpha) = \frac{a_1^2 - a_0^2}{a_1^2 + a_0^2} \tag{2}$$

for some natural numbers a_0 and a_1.
Similar for the triangles of Heron. The form of such a triangle is determined by two Q-angles. The sidelength are proportional to the sines of the angles. So only one sidelength have to be chosen.

Definition.
A Q-configuration in the plane is a set of line segments, enclosing area's. The segments are rational numbers in some unit and the area's are rational numbers in the associative square unit. The smallest Q-configuration is the Q-configuration with relatively prime numbers instead of rational numbers.

The only Q-triangles are the triangles of Pythagoras and Heron.
The smallest Q-configuration is the square, with sides and area equal to 1.

This Q-square cannot be inscribed in a Q-circle, because the lengths of the radius and the side are not both rational in the same unit.

2 Some examples of Q-configurations

1. The smalles Q-triangle is the right triangle with sidelengths 3, 4 and 5 and area 6.

2. The smallest scalene Q-triangle is the triangle with sidelength 13, 14 and 15 and area 84.

3. The smallest Q-quadrangle, inscribed in a circle, are the rightangle with sidelengths 3, 4, 3, 4 and the kite with sidelength 3, 3, 4, 4 both with area 12.

4. Rightangled $\triangle ABC$ met right $\angle C$ is a Q-triangle, when the sides are rational. In the formulas is r the radius of the inscribed circle and R the radius of the circiumcircle.

$$\angle A = \cos(\alpha) + \sin(\alpha).\imath = \frac{a_1^2 - a_0^2}{a_1^2 + a_0^2} + \frac{2a_1a_0}{a_1^2 + a_0^2}.\imath \tag{3}$$

$$a : b : c = \sin(\alpha) : \sin(\beta) : \sin(\gamma) = \frac{2a_1a_0}{a_1^2 + a_0^2} : \frac{a_1^2 - a_0^2}{a_1^2 + a_0^2} : 1 \tag{4}$$

$$a = 2a_1a_0, \quad b = a_1^2 - a_0^2, \quad c = a_1^2 + a_0^2 \tag{5}$$

$$s = \frac{1}{2}(a + b + c) = a_1(a_1 + a_0) \tag{6}$$

$$\text{Area}(\triangle ABC) = a_1a_0(a_1^2 - a_0^2) \tag{7}$$

$$r = a_0(a_1 - a_0) \tag{8}$$

$$R = \frac{a}{2\sin(\alpha)} = \frac{a_1^2 + a_0^2}{2} \tag{9}$$

5. Isoscele $\triangle ABC$, consisting of two Q-triangle from the previous example. This triangle is determined by a base angle. Let $\angle A = \angle B$ the both base angles and $\angle C$ de vertex angle. Then

$$\angle A = \angle B = \cos(\alpha) + \sin(\alpha) = \frac{a_1^2 - a_0^2}{a_1^2 + a_0^2} + \frac{2a_1a_0}{a_1^2 + a_0^2}.\imath \tag{10}$$

$$\angle C = -\frac{(a_1^2 - a_0^2)^2 - 4(a_1)^2(a_0)^2}{(a_1^2 + a_0^2)^2} + \frac{4a_1a_0(a_1^2 - a_0^2)}{(a_1^2 + a_0^2)^2}.\imath \tag{11}$$

$$a : c = \frac{2a_1a_0}{a_1^2 + a_0^2} : \frac{4a_1a_0(a_1^2 - a_0^2)}{(a_1^2 + a_0^2)^2} = (a_1^2 + a_0^2) : 2(a_1^2 - a_0^2) \tag{12}$$

$$a = b = a_1^2 + a_0^2, \quad c = 2(a_1^2 - a_0^2), \quad s = 2a_1^2 \tag{13}$$

$$\text{Area}(\triangle ABC) = 2a_1a_0(a_1^2 - a_0^2) \tag{14}$$

$$r = \frac{a_0(a_1^2 - a_0^2)}{a_1} \tag{15}$$

$$R = \frac{a_1^2 + a_0^2}{2\frac{2a_1a_0}{a_1^2+a_0^2}} = \frac{(a_1^2 + a_0^2)^2}{4a_1a_0} \tag{16}$$

6. Scalene Q-triangle ABC is determined by two Q-angles. Let

$$\angle A = \cos(\alpha) + \sin(\alpha).\imath = \frac{a_1^2 - a_0^2}{a_1^2 + a_0^2} + \frac{2a_1a_0}{a_1^2 + a_0^2}.\imath \tag{17}$$

$$\angle B = \cos(\beta) + \sin(\beta).\imath = \frac{a_2^2 - a_0^2}{a_2^2 + a_0^2} + \frac{2a_2a_0}{a_2^2 + a_0^2}.\imath \tag{18}$$

Then

$$sin(\angle C) = sin(\alpha + \beta) = \frac{2a_0(a_1 + a_2)(a_1a_2 - a_0^2)}{(a_1^2 + a_0^2)(a_2^2 + a_0^2)} \tag{19}$$

$$a : b : c = a_1(a_2^2 + a_0^2) : a_2(a_1^2 + a_0^2) : (a_1 + a_2)(a_1a_2 - a_0^2) \tag{20}$$

And for the sides and area:

$$a = a_1(a_2^2 + a_0^2), \quad b = a_2(a_1^2 + a_0^2), \quad c = (a_1 + a_2)(a_1a_2 - a_0^2) \tag{21}$$

$$s = a_1a_2(a_1 + a_2) \tag{22}$$

$$\text{Area}(\triangle ABC) = a_0a_1a_2(a_1 + a_2)(a_1a_2 - a_0^2) \tag{23}$$

$$r = a_0(a_1a_2 - a_0^2) \tag{24}$$

$$R = \frac{(a_1^2 + a_0^2)(a_2^2 + a_0^2)}{4a_0} \tag{25}$$

7. For several years stands Figure 1 on the homepage of http://b.duizendknoop.com/.
It is the configuration of an scalene triangle with altitudes intersecting at the orthocenter. How many square units is the area of $\triangle(ABC)$ in the smallest Q-configuration?

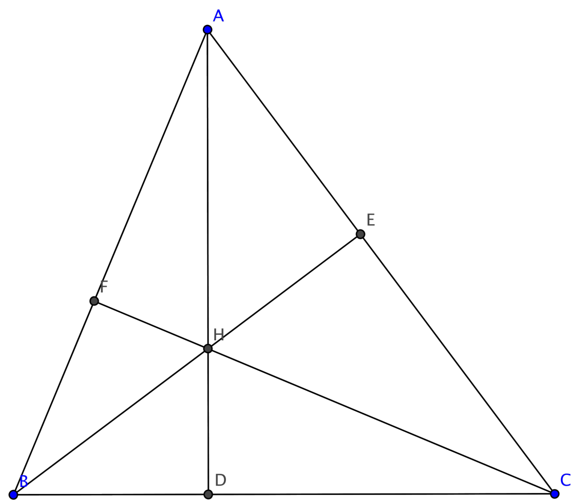

Figure 1:

3 Q-quadrangle, inscribed in a circle.

In this section some formulas are derived. For the length of sides and diagonals and area's of the inscribed Q-quadrangle and some subtriangles.

Definition.
An inscribed Q-quadrangle is defined by four points on a circle such, that the lengths of the four sides and the two diagonals in some unit and the area's of the quadrangle and each of the subtriangles in squares of the same unit are rational numbers.
In the configuration (see fig 2) of this section is $\triangle ABC$ a Q-triangle and R is the radius of the circumcircle of the triangle. D is a point on this circle such, that $\angle DBC$ is a Q-angle. From $\angle CAB = \angle CDB$ it follows that all angles in the configuration are Q-angles. And moreover the each of the triangles has the same circumradius R. Let

$$\angle ACB = \angle ADB = \angle C_2 = \angle D_1 = \cos(\alpha) + \sin(\alpha).\imath \qquad (26)$$

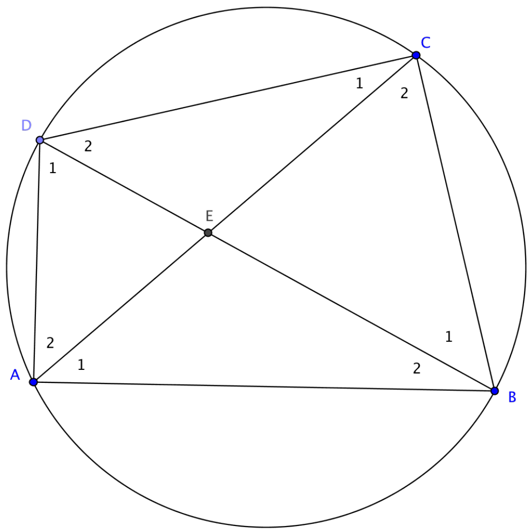

Figure 2:

$$\angle CAD = \angle CDB = \angle A_1 = \angle D_2 = \cos(\beta) + \sin(\beta).\imath \tag{27}$$

$$\angle CAD = \angle CBD = \angle A_2 = \angle B_1 = \cos(\gamma) + \sin(\gamma).\imath \tag{28}$$

with

$$\sin(\alpha) = \frac{2a_1a_0}{a_1^2 + a_0^2}, \quad \sin(\beta) = \frac{2a_2a_0}{a_2^2 + a_0^2}, \quad \sin(\gamma) = \frac{2a_3a_0}{a_3^2 + a_0^2} \tag{29}$$

After some computation:

$$\sin(B_{12}) = \sin(180^0 - A_1 - C_2) = \frac{2a_0(a_1 + a_2)(a_1a_2 - a_0^2)}{(a_1^2 + a_0^2)(a_2^2 + a_0^2)} \tag{30}$$

$$\cos(B_{12}) = -\cos(180^0 - A_1 - C_2) = \frac{(a_1a_2 - a_0^2)^2 - (a_0(a_1 + a_2))^2}{(a_1^2 + a_0^2)(a_2^2 + a_0^2)} \tag{31}$$

$$\sin(C_{12}) = \sin(180^0 - B_1 - D_2) = \frac{2a_0(a_2 + a_3)(a_2a_3 - a_0^2)}{(a_2^2 + a_0^2)(a_2^3 + a_0^2)} \tag{32}$$

$$\sin(B_2) = \sin(180^0 - C_2 - A_! - A_2) = \frac{2a_0(a_0^2(a_1 + a_2 + a_3) - a_1a_2a_3)(a_0^2 - a_1a_2 - a_2a_3 - a_3a_1)}{(a_1^2 + a_0^2)(a_2^2 + a_0^2)(a_3^2 + a_0^2)} \tag{33}$$

After multiplication with $\frac{(a_1^2+a_0^2)(a_2^2+a_0^2)(a_3^2+a_0^2)}{2a_0}$ to get whole numbers and using the sine rule:

$$AB = a_1(a_2^2 + a_0^2)(a_3^2 + a_0^2) \tag{34}$$

$$BC = a_2(a_1^2 + a_0^2)(a_3^2 + a_0^2) \tag{35}$$

$$CD = a_3(a_1^2 + a_0^2)(a_2^2 + a_0^2) \tag{36}$$

$$AD = (a_1a_2a_3 - a_0^2(a_1 + a_2 + a_3))(a_1a_2 + a_2a_3 + a_3 - a_1a_0^2) \tag{37}$$

$$AC = (a_1 + a_2)(a_1a_2 - a_0^2)(a_3^2 + a_0^2) \tag{38}$$

$$BD = (a_2 + a_3)(a_2a_3 - a_0^2)(a_1^2 + a_0^2) \tag{39}$$

$$R = \frac{AB}{2\sin(C_2)} = \frac{(a_1^2 + a_0^2)(a_2^2 + a_0^2)(a_3^2 + a_0^2)}{4a_0} \tag{40}$$

Proposition.
With the above formulas for the sides of the quadrangle is the formula for the area of the quadrangle

$$\text{Area}(ABCD) = a_0 \prod_{cycl}^{1,2,3} (a_i + a_j)(a_ia_j - a_0^2) \tag{41}$$

Proof.
This proof uses the semiperimeter s of the Q-quadrangle $ABCD$ and the formula of Brahmagupta.

$$s = a_0^2(a_0^2(a_1 + a_2 + a_3) - a_1a_2a_3) + a_1a_2a_3(a_1a_2 + a_2a_3 + a_3a_1 - a_0^2) \tag{42}$$

$$s - AB = (a_2 + a_3)(a_1a_2 - a_0^2)(a_3a_1 - a_0^2) \tag{43}$$

$$s - BC = (a_3 + a_1)(a_3a_1 - a_0^2)(a_2a_3 - a_0^2) \tag{44}$$

$$s - CD = (a_1 + a_2)(a_2a_3 - a_0^2)(a_1a_2 - a_0^2) \tag{45}$$

$$s - DA = a_0^2(a_1 + a_2)(a_2 + a_3)(a_3 + a_1) \tag{46}$$

After substitution to Brahmagupta's formel we get

$$\text{Area}(ABCD) = \sqrt{(s - AB)(s - BC)(s - CD)(s - DA)} \tag{47}$$

$$\text{Area}(ABCD) = a_0 \prod_{cycl}^{1,2,3} (a_i + a_j)(a_ia_j - a_0^2) \tag{48}$$

4 Almost regular Q-triangle.

The regular triangle has angles of 60^0. So there exists no regular Q-triangle.
Definition.
An almost regular Q-triangle is a Q-quadrangle with three equal sides and the fourth side much smaller than the three equal sides. (See fig 3) Let

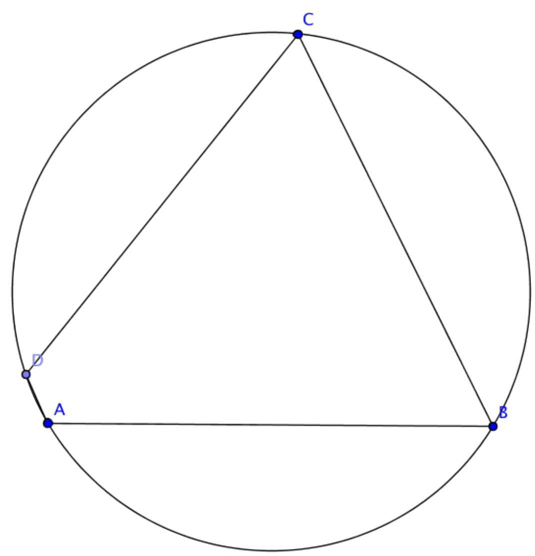

Figure 3:

$ABCD$ an inscribed Q-quadrangle with $AB = BC = CD$ en $AD \ll AB$ and $\angle CAB = \cos(\alpha) + \sin(\alpha).\imath$ met $\sin(\alpha) = \frac{2a_1a_0}{a_1^2+a_0^2}$. Then using the prededing section and multiplication with $4a_0$ we get:

$$AB = BC = CD = 4a_1a_0(a_1^2 + a_0^2)^2 \tag{49}$$

$$AD = 4a_1a_0(3a_0^2 - a_1^2)(a_0^2 - 3a_1^2) = 4a_1a_0(3a_1^2 - a_0^2)(a_1^2 - 3a_0^2) \tag{50}$$

$$AC = BD = 8a_1a_0(a_1^2 - a_0^2)(a_1^2 + a_0^2) = 8a_1a_0(a_1^4 - a_0^4) \tag{51}$$

The radius of the circumcircle is

$$R = (a_1^2 + a_0^2)^3 \tag{52}$$

An estimation of the difference between an almost regular Q-triangle and an equilateral triangle.
We need an angle of almost 60^0. This angle is twice $30^0 = \frac{1}{2}\sqrt{3} + \frac{1}{2}$.
The continued fraction $\sqrt{3} = (1; 1, 2, 1, 2, ...)$ generates for $\frac{a_1}{a_0}$ the the converging sequence

$$\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \ldots \tag{53}$$

In the following table you can find the beginning of the sequence of triangles, who are generated by this sequence. Remark. The minus signs occur, when

a_1	a_0	AB	AD	AD/AB
1	1	16	-16	-1
2	1	200	88	0.44
5	3	69360	-7920	-0,114
7	4	473200	14672	0.03101
19	11	194222864	-1608464	-0.0082815
26	15	1266409560	2812680	0.002220988
71	41	526137446896	-313037296	-0.00059497
97	56	3419487999200	545177248	0.000159432

Table 1:

the sides AB and CD meet in an interior point of the circle.

5 Inscribed Q-polygons

In this section some properties about inscribed Q-polygons. We use the formula of de Moivre to construct an almost regular Q-n-polygon.
Let $\triangle ABC$ be a Q-triangle, inscribed in a cirkel and let P be a point on the circumcircle such, that $\angle PAB$ is a Q-angle. Then $ABCP$ is a Q-quadrangle. By adding ponts $Q, R, S, \ldots$ we find a Q-polygon $ABC...PQR\ldots$. A corollary of the property in Euclides book III-21 is that the diagonals of a Q-polygon make Q-angles with each other. Using the sine-rule it is easy to proof that the diagonals devide each other in rational line-segments. So we have the following

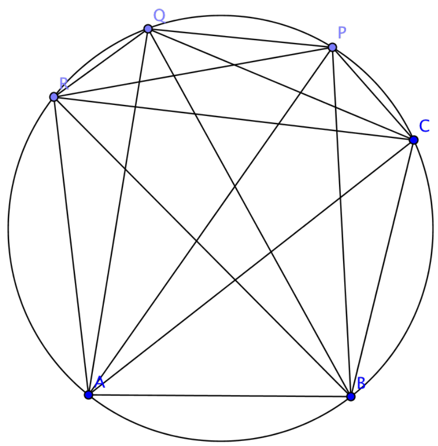

Figure 4:

Proposition.
The configuration of a Q-polygon, iscribed in a circle, with all its diagonals is a Q-configuration.
Similar as in the preceeding section we define almost regular Q-n-polygons $A_0A_1...A_n$. Using the formula of de Moivre follows the length of A_nA_0.

$$\cos(n\alpha) + \sin(n\alpha).\imath = (\cos(\alpha) + \sin(\alpha).\imath)^n \tag{54}$$

We have for n is even:

$$\sin(n\alpha) = \cos(\alpha)\sin(\alpha)\sum_{i=1}^{\frac{1}{2}n}\left(\sum_{k=1}^{i}\binom{n}{2k-1}\binom{\frac{1}{2}n-k}{i-k}.(-\sin^2(\alpha))^{i-1}\right) \tag{55}$$

and for n is odd:

$$\sin(n\alpha) = \sin(\alpha)\sum_{i=1}^{\frac{1}{2}(n+1)}\left(\sum_{k=1}^{i}\binom{n}{2k-1}\binom{\frac{1}{2}(n+1)-k}{i-k}.(-\sin^2(\alpha))^{i-1}\right) \tag{56}$$

As an example we end with an approximation of $\frac{A_5A_0}{A_1A_0}$.

$$\frac{A_5A_0}{A_1A_0} = \frac{\sin(5\alpha)}{\sin(\alpha)} = 5 - 20\sin^2(\alpha) + 16\sin^4(\alpha) \tag{57}$$

We take the fraction $\frac{355}{113}$ by Metius / Zu Chongzi to get the approximation $\alpha = \frac{355}{5.113} = \frac{71}{113}$. Then $\sin(\alpha) \approx \frac{71}{113} - \frac{1}{6}\left(\frac{71}{113}\right)^3$ gives

$$\frac{A_5A_0}{A_1A_0} = 5 - 20\sin^2(\alpha) + 16\sin^4(\alpha) = \frac{2987270259416040848540881}{3510963711155266258028573661} \tag{58}$$

A Goniometrical (basic)formules

$$(\cos(\alpha) + \sin(\alpha).\imath)(\cos(\beta) + \sin(\beta).\imath) = \cos(\alpha+\beta) + \sin(\alpha+\beta).\imath$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha+\beta+\gamma) = \sin(\alpha)\cos(\beta)\cos(\gamma) + \sin(\beta)\cos(\gamma)\cos(\alpha) +$$

$$+\sin(\gamma)\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\sin(\gamma)$$

$$\cos(\alpha+\beta+\gamma) = \cos(\alpha)\cos(\beta)\cos(\gamma) +$$

$$-\cos(\alpha)\sin(\beta)\sin(\gamma) - \cos(\beta)\sin(\gamma)\sin(\alpha) - \cos(\gamma)\sin(\alpha)\sin(\beta)$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$

$$\sin(3\alpha) = 3\sin(\alpha) - 4\sin^3(\alpha)$$

$$\cos(3\alpha) = 4\cos^3(\alpha) - 3\cos(\alpha)$$

$$\sin(180^0 - 3\alpha) = 3\sin(\alpha) - 4\sin^3(\alpha)$$

$$\cos(180^0 - 3\alpha) = 3\cos(\alpha) - 4\cos^3(\alpha)$$

$$\sin(180^0 - 3\alpha) = 3\frac{2ab}{a^2+b^2} - 4\left(\frac{2ab}{a^2+b^2}\right)^3 = \frac{2ab(3a^2-b^2)(a^2-3b^2)}{(a^2+b^2)^3}$$

$$\cos(180^0 - 3\alpha) = 3\frac{a^2-b^2}{a^2+b^2} - 4\left(\frac{a^2-b^2}{a^2+b^2}\right)^3 = \frac{(a^2-b^2)(14a^2b^2-a^4-b^4)}{(a^2+b^2)^3}$$